

The fermionic contribution to the spectrum of the area operator in nonperturbative quantum gravity

Merced Montesinos^{a,b*} and Carlo Rovelli^{a†}

^aDepartment of Physics and Astronomy, University of Pittsburgh,
Pittsburgh, PA 15260, USA.

^bDepartamento de Física, Centro de Investigación y de Estudios Avanzados del I.P.N.,
Av. I.P.N. No. 2508, 07000 Ciudad de México, México.

Abstract

The role of fermionic matter in the spectrum of the area operator is analysed using the Baez–Krasnov framework for quantum fermions and gravity. The result is that the fermionic contribution to the area of a surface S is equivalent to the contribution of purely gravitational spin network’s edges tangent to S . Therefore, the spectrum of the area operator is the same as in the pure gravity case.

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Loop quantum gravity [1], the nonperturbative approach to quantum gravity, is nowadays a mathematically well-defined theory with a powerful *predictive* character (see [2] for a recent review). The theory is based on the Hamiltonian formulation of general relativity due to Ashtekar [3] which, as was shown in [4], is the ADM formulation of the (self–dual sector of the) Plebanski action [5]. At present, the theory is usually formulated in terms of the real $SU(2)$ Ashtekar connection, whose use has been advocated by Barbero [6], and which can be obtained through a canonical transformation from the original complex Ashtekar variables. Amongst the most striking results of loop quantum gravity are the spectra of the area and volume operators [7–9], and the computation of the entropy of black holes [10]. These results point to the existence of discrete aspects of spacetime at the Planck length $l_P = \sqrt{G\hbar/c^3}$.

The research in loop quantum gravity is presently developing along three main directions. The first of these focuses on the physics of black holes [10,11]. The second deals with the the Hamiltonian constraint [12,13] and with Feynman–type formulations [14] of the quantum dynamics. The third studies the coupling of matter fields to quantum gravity. For instance, in [15] the contribution of the quantum states to the fermionic mass has been studied, while the possibility of a quantum-gravity induced vanishing of the cosmological constant has

*E–mail: merced@fis.cinvestav.mx

†E–mail: roveli@pitt.edu

been explored in [16]. Although these results are preliminary, they indicate that unexpected phenomena may result from the coupling between quantum matter fields and the non-perturbative quantum gravitational field. This paper deals with this third direction. In particular, our aim is to study the modifications of the spectrum of the area operator due to the presence of a fermion field.

Our main result is that the spectrum of the area is not altered by the presence of fermions. This result was already anticipated in [8], on the basis of general considerations on the form of gravity-matter theories. Here, we verify it explicitly within a specific fermionic model.

The model we consider is the Einstein–Weyl theory. We follow the approach developed in [17], which has its roots in the previous works of [18], and we take (as in [17]) $SU(2)$ as the relevant internal group. For our purposes, the key difference from other matter couplings is that in the case of the spin-1/2 field there is a matter contribution to the gravitational Gauss constraint. The other matter fields do not have this property. This difference is the reason for the nontrivial interplay between the area and the fermions.

The area operator \hat{A}_S associated to a two-dimensional surface S is well understood [7,8]. Let us recall its definition. We take S to be an open surface with boundaries (we will comment on closed surfaces later on). The operator can be written as

$$\hat{A}_S = 8\pi\beta l_P^2 \sum_{v \in S} \sqrt{\hat{A}_{S,v}^2}, \quad (1)$$

where β is the Immirzi parameter [20] and $\hat{A}_{S,v}^2$ is the vertex area operator. This operator acts on the vertices v of the spin network states [19], lying on the surface S . It can be written as

$$\hat{A}_{S,v}^2 = \frac{1}{2} \epsilon^{AC} \epsilon^{BD} \left(\hat{J}_{AB}^{(d)} - \hat{J}_{AB}^{(u)} \right) \left(\hat{J}_{CD}^{(d)} - \hat{J}_{CD}^{(u)} \right). \quad (2)$$

Here $\hat{J}^{(u)} = \sum_i \hat{J}_i^{(u) \text{ outgoing}} + \sum_j \hat{J}_j^{(u) \text{ incoming}}$ and $\hat{J}^{(d)} = \sum_i \hat{J}_i^{(d) \text{ outgoing}} + \sum_j \hat{J}_j^{(d) \text{ incoming}}$ are the (symmetric) ‘link operators’[‡] associated with the edges of the spin network on the two sides (‘up’ and ‘down’) of the surface S ; the sums run over the edges γ_i , ‘outgoing’ and ‘incoming’, at the vertex. For a particular edge γ_j , the action of the operator \hat{J}_{AB} on a cylindrical function $\Psi_{\Gamma,f}(A)$ (see [19]) is given by

[‡]These operators are the ‘spinorial version’ of the Ashtekar–Lewandowski (see [8]) angular momentum operators.

$$\hat{J}_{j\ AB}\Psi_{\Gamma,f}(A) := \begin{cases} +\frac{i}{2}\epsilon_{C(A}U_B)^D[\gamma_j, A]\frac{\partial f}{\partial U_C^D[\gamma_j, A]}, & \text{if } \gamma_j \text{ is 'outgoing'}, \\ -\frac{i}{2}U_{C(A}[\gamma_j, A]\delta_B)^D\frac{\partial f}{\partial U_C^D[\gamma_j, A]}, & \text{if } \gamma_j \text{ is 'incoming'}, \end{cases} \quad (3)$$

The above formulae follow directly from the definition of the area operator, and are valid with as well as without fermions.

In the pure gravity case, the gravitational Gauss constraint $\hat{G}[N]^{Einstein}\Psi_{\Gamma,f}(A) = 0$, implies that at every vertex of the spin networks lying on the surface S the following condition holds

$$\hat{J}_{AB}^{(u)} + \hat{J}_{AB}^{(d)} + \hat{J}_{AB}^{(t)} = 0, \quad (4)$$

where $\hat{J}_{AB}^{(t)}$ is the sum of the link operators associated to the edges that exit v tangentially with respect to S (neither ‘up’ nor ‘down’). This fact allows us to rewrite (2) as

$$\hat{A}_{S,v}^2 = \frac{1}{2}\epsilon^{AC}\epsilon^{BD}\left(2\hat{J}_{AB}^{(u)}\hat{J}_{CD}^{(u)} + 2\hat{J}_{AB}^{(d)}\hat{J}_{CD}^{(d)} - \hat{J}_{AB}^{(t)}\hat{J}_{CD}^{(t)}\right). \quad (5)$$

Since the three terms of the sum can be diagonalized simultaneously, the purely gravitational spectrum of the area

$$A_S = 8\pi\beta l_P^2 \sum_v \sqrt{\frac{1}{2}j_v^u(j_v^u + 1) + \frac{1}{2}j_v^d(j_v^d + 1) - \frac{1}{4}j_v^t(j_v^t + 1)}, \quad (6)$$

follows easily. Here j_v^u , j_v^d and j_v^t are the total spins of the upgoing, downgoing and tangential edges of the vertex v .

Let us come now to the Einstein–Weyl theory. Let η^A and $\tilde{\pi}_B$ be the fermionic field and its conjugate momentum, respectively. The purely gravitational Gauss constraint $G_{AB}^{Einstein}$ becomes [21]

$$\mathcal{G}_{AB}(x) := G_{AB}^{Einstein}(x) + \eta_{(A}(x)\tilde{\pi}_{B)}(x). \quad (7)$$

A straightforward calculation shows that the quantum version of this constraint on a fermionic cylindrical function $\Psi_{\Gamma,f}(A, \eta)$ [17] implies that the following condition must hold at every vertex of the spin networks which lies on the surface S

$$\hat{J}_{AB}^{(u)} + \hat{J}_{AB}^{(d)} + \hat{J}_{AB}^{(t)} + \hat{S}_{AB} = 0, \quad (8)$$

where

$$\hat{S}_{AB}(v) \cdot \Psi_{\Gamma,f}(A, \eta) = \begin{cases} 0 & , \text{if no fermions sit in the vertex } v, \\ -\frac{i}{2}\eta_{(A}\frac{\partial^L}{\partial \eta^{B)}(v)}\Psi_{\Gamma,f}(A, \eta) & , \text{if fermions sit in the vertex } v. \end{cases} \quad (9)$$

(The superscript ‘L’ stands for left derivatives [22]). Taking into account equation (8), the formula for the square of the vertex area operator (2) can be written as

$$\hat{A}_{S,v}^2 = \frac{1}{2} \epsilon^{AC} \epsilon^{BD} \left[2\hat{J}_{AB}^{(u)} \hat{J}_{CD}^{(u)} + 2\hat{J}_{AB}^{(d)} \hat{J}_{CD}^{(d)} - \left(\hat{J}_{AB}^{(t)} + \hat{S}_{AB} \right) \cdot \left(\hat{J}_{CD}^{(t)} + \hat{S}_{CD} \right) \right]. \quad (10)$$

We analyze here this operator, i.e. its action on the fermionic spin network states [17].

First of all, notice that owing to the Grassman property of the fermionic field, there are restrictions on the number of fermionic fields which can sit at a given vertex of the fermionic spin networks. In general, for gauge-invariant fermionic spin networks, there can be only zero, one or two fermions at each vertex of the fermionic spin networks. That is, the quantum state can only be independent, linearly dependent, or quadratically dependent, from the value of the fermion field in a given vertex. Therefore, there are three cases in the analysis for the action of the $\hat{A}_{S,v}^2$ operator.

I. No fermions at the vertex. From equations (9), the meaning of (10) is clear: if there are no fermionic excitations at a given vertex of the spin network, \hat{S}_{AB} vanishes, and the action of $\hat{A}_{S,v}^2$ operator reduces to the pure gravitational one. This result comes from the fact that the action of the vertex area operator is ‘local’, i.e. the contribution of fermionic spin networks and pure gravitational spin networks is exactly the same for every vertex which has no fermionic excitations.

II. Two fermions at the vertex. When there are two fermions in a given vertex of the fermionic spin network, a straightforward calculation shows that the action of the operator $-\frac{1}{2} \epsilon^{AC} \epsilon^{BD} \left(\hat{J}_{AB}^{(t)} + \hat{S}_{AB} \right) \cdot \left(\hat{J}_{CD}^{(t)} + \hat{S}_{CD} \right)$ in (10) on the two fermions $\eta^E \eta^F$ which sit at the vertex vanishes. This result follows from the anticommuting properties of the spinor field, together with the symmetry of the $\hat{J}_{AB}^{(t)}$ operator. Therefore the contribution to $\hat{A}_{S,v}^2$, in this case, comes only from the edges of the spin network, i.e. even though there are two fermions in the vertex, the action of the fermionic operator \hat{S}_{AB} is missing. Consequently, the contribution in this situation is as in (10) with $\hat{S}_{AB} = 0$.

III. One fermion at the vertex. It is instructive to consider first a simplified case (a ‘one-side’ vertex). Assume that a vertex has no ‘down’ nor tangential links, but only ‘up’ links. Thus $\hat{J}_{AB}^{(u)} \neq 0$, $\hat{J}_{AB}^{(d)} = 0$, $\hat{J}_{AB}^{(t)} = 0$. In the pure gravitational case, such a vertex has

vanishing contribution. Assume a fermion sits in the vertex. We have

$$2\epsilon^{AC}\epsilon^{BD}\hat{S}_{AB}\hat{S}_{CD}\eta^E = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\eta^E, \quad (11)$$

(that is, the fermion has spin $\frac{1}{2}$). Using this and (8) we have that (10), reduces to

$$\hat{A}_{S,v}^2 = \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right). \quad (12)$$

Therefore the fermion contributes to the area formula (6). The contribution is the same as the one from a tangential link with spin $j = \frac{1}{2}$. As a second instructive example, consider the case of a vertex with no tangential edges and a single fermion. Using (11), we obtain the contribution to the square of the vertex area operator

$$\hat{A}_{S,v}^2 = \frac{1}{2}\epsilon^{AC}\epsilon^{BD}\left[2\hat{J}_{AB}^{(u)}\hat{J}_{CD}^{(u)} + 2\hat{J}_{AB}^{(d)}\hat{J}_{CD}^{(d)}\right] - \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right). \quad (13)$$

Again, the fermionic contribution is the same as the contribution of a tangential link with spin $\frac{1}{2}$. In general, in fact, it is easy to see from the fact that the ‘tangential’ angular momentum $\hat{J}_{AB}^{(t)}$ and the fermion’s ‘spin’ \hat{S}_{AB} in (10), appear only in the combination $\hat{J}_{AB}^{(t)} + \hat{S}_{AB}$, that the two cases above illustrate the general situation: the effect on the area operator of the presence of a fermion is the same as the effect of the presence of an additional tangential edge with spin $j = \frac{1}{2}$.

In conclusion, we have studied the contribution of the spin-1/2 matter field (within the framework of the Einstein–Weyl theory) to the spectrum of the area operator. Our result is that the effect of the fermions at a given vertex of the fermionic spin network–lying on the surface S – is equivalent to the effect of a tangential spin network edge with spin $j = \frac{1}{2}$ at such vertex. Therefore, the spectrum of the area operator does not change if fermionic matter is present, in spite of the fact that the quantum states do.

Our result can be understood intuitively as follows. The area operator ‘sees’ only the edges of the spin network that are not tangential to the surface (see Eq.(2)). This is due to the fact that the loop states (of which the edges of the spin network are composed) carry *transversal* quanta of area, contributing only to the area of surfaces that are not tangent to the loop itself. See [23] for a discussion of the relevant geometry. Thus, from the point of view of the area operator, a tangential spin network edge is like an edge ‘moving out of the manifold’ carrying angular momentum with itself. Now, precisely the same is

true for fermions: in fact, it has been repeatedly noticed that in loop quantum gravity a fermion behaves precisely as a gravitational loop ‘continuing out of the manifold’. In [18], it was noticed that the dynamics of a fermion is the same as the dynamics generated by the *purely gravitational* quantum hamiltonian constraint on the end point of loop states ‘cut open’, more precisely, generated by the shift operator [1]. In [25], it was even suggested to interpret this very surprising fact in terms of John Wheeler’s ideas of particles as wormholes: a fermion is the point in which a gravitational line of flux plunges into a small wormhole. In [24] the similarity between fermions and end points of loop states was extended to the loop’s end points on boundaries of the manifold. Here, we have shown that a fermion behaves as ‘a gravitational line of flux continuing out of the manifold’ also as far as the area is concerned.

The case of a *closed* surface is slightly different, due to the ‘fermion number conservation’ discussed in detail in [8]. This is a restriction of the spectrum which can be easily understood using the ‘old’ overcomplete loop basis instead of the spin network basis (see [19]): every loop that enters the closed surface must exit it, so that a closed surface is always crossed by an *even* number of loops. Since a fermion plays the role of end-point of a loop, this restriction disappears in the presence of fermions. Therefore, the theory with fermions differs from the pure gravity theory in the fact that, in the presence of fermions, the spectrum of the area of a closed surface is the same as the spectrum of the area of an open surface [§].

We notice that since the classical limit of the area observable is presumably recovered in the $j \rightarrow \infty$ limit, the contribution of the fermionic spin network states, which is given by spin $j = 1/2$ terms only, can be seen as a *pure quantum* effect. On the other hand, we recall that the contribution of the spin $j = 1/2$ terms plays a dominant role for the value of the entropy of black holes [10]; therefore the results presented here might have implications for black hole entropy.

The case of the Einstein–Dirac [21] system follows the same lines as the present case if one follows (as here) the formalism of [17]; the only difference with respect to the present case is the contribution of two types of fermionic operators, associated with the two two-component fermionic fields. From the general results in [8], the same result also holds for a general class of matter fields including Thiemann’s fermions [26]. On the other hand, the role of the fermionic matter in the spectrum of the volume operator [9] is not yet known,

[§]We thank Jerzy Lewandowski for this observation.

and deserves to be studied.

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